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Entanglement detection beyond the CCNR criterion for infinite-dimensions

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In terms of the relation between the state and its reduced states, we obtain two inequalities which are valid for all separable states in infinite-dimensional bipartite quantum systems. One of them provides an entanglement criterion which is strictly stronger than the computable cross-norm or realignment (CCNR) criterion.

quantum state, entanglement, computable cross-norm or realignment criterion, infinite-dimensional quantum systems

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Quantum entanglement has been subjected to intensive studies in connection with quantum information theory and quantum communication theory [1]. One basic problem for quantum entanglement is to find a proper criterion to determine whether a given state of a composite system is entangled or not [2–18]. Although considerable progress has been achieved in this field, this problem is not fully explored yet except for the case of $2\otimes 2$ and $2\otimes 3$ systems [2, 3, 19].

By definition, a bipartite state ρ acting on a separable complex Hilbert space $H = H_A \otimes H_B$ is called *separable* if it can be written as

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad \sum_i p_i = 1,\ p_i \geqslant 0 \tag{1}$$

or it is a limit of the states of the above form under the trace norm topology [20], where ρ_i^A and ρ_i^B are (pure) states on the subsystems associated to the Hilbert spaces H_A and H_B, respectively. A state that is not separable is said to be entangled. Particularly, if a state can be represented in the form as in eq. (1), it is called countably separable [21]. Observing that, for finite-dimensional systems, all separable states are finitely separable. However, there do exist separable states which are not countably separable in infinite-dimensional systems [21]. Thus it is interesting to study the question that what kind of criteria are still valid or can be extended to infinite-dimensional systems. This question has been attacked by several authors (for example, [10–15], [19, 21, 22] and the references therein).

For finite-dimensional systems, a very elegant criterion for the separability is the so-called computable cross-norm or realignment (CCNR) criterion proposed by Rudolph in [23] and Chen and Wu in [24]. The CCNR criterion states that if ρ is a separable state on $H_A \otimes H_B$ with $\dim H_A \otimes H_B < +\infty$, then the trace norm $\|\rho^R\|_{\rm Tr}$ of the realignment matrix ρ^R of ρ is not greater than 1. By exploring the relation between the state and its reduced states, Zhang et al. [25] investigated a criterion beyond the CCNR criterion. It is showed in [25] that, if a state acting on $H_A \otimes H_B$ with $\dim H_A \otimes H_B < +\infty$ is separable, then

$$\|(\rho - \rho_A \otimes \rho_B)^R\|_{\rm Tr} \leqslant \sqrt{[1 - {\rm Tr}(\rho_A^2)][1 - {\rm Tr}(\rho_B^2)]} \tag{2}$$

and

$$\|(\rho - \rho_A \otimes \rho_B)^{T_B}\|_{\rm Tr} \leqslant 2\sqrt{[1 - {\rm Tr}(\rho_A^2)][1 - {\rm Tr}(\rho_B^2)]}. \tag{3}$$

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Here, C^R denotes the realignment matrix of the block matrix $C = [C_{ij}]_{N_A\times N_A}$ with C_{ij}s are $N_B \times N_B$ complex matrices with $\dim H_A = N_A$ and $\dim H_B = N_B$; $\|\cdot\|_{\rm Tr}$ denotes the trace norm and C^{T_B} denotes the partial transposition of C with respect to the subsystem B. The inequality (2) provides a criterion which is stronger than the CCNR criterion [25] (namely, any entangled states that detected by the CCNR criterion can be detected by the inequality (2) and there exist some entangled states that can be detected by inequality (2) while they cannot be recognized by the CCNR criterion).

Very recently, we established the realignment operation and CCNR criterion for infinite-dimensional bipartite systems [26, 27]. We generalized the realignment operation of matrices to Hilbert-Schmidt operators on infinite-dimensional Hilbert space $H_A \otimes H_B$ in [26] and showed that $\|\rho^R\|_{\rm Tr} \leqslant 1$ whenever ρ is a separable state acting on $H_A \otimes H_B$. The aim of this paper is to establish the analogous inequalities as (2) and (3) for infinite-dimensional case. As one might expect, we show that the obtained inequality criterion is stronger than the CCNR criterion proposed in [26]. Furthermore, it can detect some PPT entangled states (i.e. the entangled states with positive partial transposition) which cannot be detected by the CCNR criterion. It should be pointed out that the corresponding inequalities for infinite-dimensional case can not be derived straightforwardly from that of the finite-dimensional case. The situations grow more complicated in the case of infinite-dimensional case.

In detail, this paper is organized as follows. In Section 1 we propose some properties of the reduced density operators for both finite- and infinite-dimensional bipartite systems. We show that the reduced states stand close to each other whenever the composite states are closed to each other. Then in Section 2 we propose a practical criterion based on $\rho - \rho_A \otimes \rho_B$. The obtained criterion is strictly stronger than the CCNR criterion. Section 3 is a short conclusion.

Throughout the paper, we use the bra-ket notations. $\langle\cdot|\cdot\rangle$ stands for the inner product in the given Hilbert spaces. The set of all (bounded linear) operators on a Hilbert space H is denoted by $\mathcal{B}(H)$, the set of all trace class operators on H is denoted by $\mathcal{T}(H)$ and the space consisting of all Schatten-p class operators on H is denoted by $C_p(H)$. $A \in \mathcal{B}(H)$ is self-adjoint if $A^\dagger = A$ ($A^\dagger$ stands for the adjoint operator of A); A is said to be positive, denoted by $A \geqslant 0$, if $A^\dagger = A$ and $\langle\psi|A|\psi\rangle \geqslant 0$ for all $|\psi\rangle \in H$. A^T stands for the transposition of the operator A. By $\mathcal{S}(H_A)$, $\mathcal{S}(H_B)$ and $\mathcal{S}(H_A \otimes H_B)$ we denote the sets of all states acting on H_A, H_B and $H_A \otimes H_B$, respectively. By $\mathcal{S}_{\rm sep}(H_A \otimes H_B)$ we denote the set of all separable states in $\mathcal{S}(H_A \otimes H_B)$. We fix in the "local state spaces" H_A and H_B orthonormal bases $\{|m\rangle\}_{m=1}^{N_A}$ and $\{|\mu\rangle\}_{\mu=1}^{N_B}$, respectively, where $\dim H_A = N_A$ and $\dim H_B = N_B$ ($N_{A/B}$ may be $+\infty$) (note that we use Latin indices for the subsystem A and the Greek indices for the subsystem B). The partial transposition of $\rho \in \mathcal{S}(H_A \otimes H_B)$ with respect to the subsystem B (resp. A) is denoted by ρ^{T_B} (resp. ρ^{T_A}), that is, $\rho^{T_B} = (I_A \otimes \mathbf{T})\rho$ (resp. $\rho^{T_A} = (\mathbf{T} \otimes I_B)\rho$), where $\mathbf{T}$ is the map of taking transpose, i.e. $\mathbf{T}C = C^T$, with respect to a given orthonormal basis.

1 The reduced density operators

To describe subsystems of a composite system, one needs the reduced density operators. It is so useful as to be virtually indispensable in the analysis of composite systems [1]. In this section, we discuss some properties about the reduced density operators.

Let H_A and H_B be complex Hilbert spaces with $\dim H_A \otimes H_B = +\infty$, $\rho = |\psi\rangle\langle\psi| \in \mathcal{S}(H_A \otimes H_B)$ be a pure state with unit vector $|\psi\rangle = \sum_{m,\mu} d_{m\mu}|m\rangle|\mu\rangle \in H_A \otimes H_B$. It is clear that $D_\psi = (d_{m\mu})$ can be regarded as an operator from H_B into H_A and it is a Hilbert-Schmidt class operator with the Hilbert-Schmidt norm $\|D_\psi\|_2 = \||\psi\rangle\|$. Under the given bases, we have

$$\begin{aligned}\rho_A &= \mathrm{Tr}_B(\rho) = (I_A \otimes \mathbf{Tr})\rho \\ &= (I_A \otimes \mathbf{Tr})(\sum_{m,\mu,n,\nu} d_{m\mu}\bar{d}_{n\nu}|m\rangle\langle n| \otimes |\mu\rangle\langle\nu|) \\ &= \sum_{m,\mu,n,\nu} d_{m\mu}\bar{d}_{n\nu}\mathrm{Tr}(|\mu\rangle\langle\nu|)|m\rangle\langle n| \\ &= \sum_{m,n,\mu} d_{m\mu}\bar{d}_{n\mu}|m\rangle\langle n| \\ &= \sum_{m,n}(\sum_{\mu} d_{m\mu}\bar{d}_{n\mu})|m\rangle\langle n| = DD^\dagger.\end{aligned}$$

Similarly, $\mathrm{Tr}_A(\rho) = (\mathbf{Tr} \otimes I_B)\rho = \rho_B = D^\dagger D$. For any mixed state $\rho \in \mathcal{S}(H_A \otimes H_B)$, let

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \quad |\psi_i\rangle \in H_A \otimes H_B,\ p_i > 0,\ \sum_i p_i = 1,$$

be the spectral decomposition. Write $|\psi_i\rangle = \sum_{m,\mu} d_{m\mu}^{(i)}|m\rangle|\mu\rangle$ and $D_i = (d_{m\mu}^{(i)})$. It turns out that

$$\rho_A = \sum_i p_i D_i D_i^\dagger, \quad \rho_B = \sum_i p_i D_i^\dagger D_i. \tag{4}$$

For the finite-dimensional case, also see [28].

If $\rho, \varrho \in \mathcal{S}(H_A \otimes H_B)$, and ρ stands close to ϱ, then, what about the distance between $\rho_{A/B}$ and $\varrho_{A/B}$? The following result is an answer to this question.

Proposition 1. Let H_A and H_B be complex separable Hilbert spaces with $\dim H_A \otimes H_B \leqslant +\infty$, $\rho, \rho_k \in \mathcal{S}(H_A \otimes H_B)$, $k = 1, 2, \ldots$ and $\lim_k \rho_k = \rho$ in trace norm. Then

$$\lim_{k\to\infty} \rho_{A(k)} = \rho_A \quad \text{and} \quad \lim_{k\to\infty} \rho_{B(k)} = \rho_B, \tag{5}$$

in trace norm, where $\rho_{A(k)} = \mathrm{Tr}_B(\rho_k)$ and $\rho_{B(k)} = \mathrm{Tr}_A(\rho_k)$.

Proof. Take orthonormal bases $\{|m\rangle\}_{m=1}^{N_A}$ and $\{|\mu\rangle\}_{\mu=1}^{N_B}$ of H_A and H_B, respectively. With respect to theses bases, we can write ρ_k and ρ in the matrix form $\rho_k = (\sigma_{mn}^{(k)})$ and $\rho = (\sigma_{mn})$, where $\sigma_{mn}^{(k)}, \sigma_{mn} \in \mathcal{T}(H_B)$. Then $\rho_{A(k)} = (\mathrm{Tr}(\sigma_{mn}^{(k)}))$ and $\rho_A = (\mathrm{Tr}(\sigma_{mn}))$. Since $\rho_k \to \rho$ as $k \to \infty$ under the trace norm topology, we have $\sigma_{mn}^{(k)} \to \rho_{mn}$ as $k \to \infty$ under the trace norm topology for each (m, n)-entry. Hence $\mathrm{Tr}(\sigma_{mn}^{(k)}) \to \mathrm{Tr}(\sigma_{mn})$ for any m, n, that is, $\rho_{A(k)}$ converges to ρ_A entry-wise. Note that $\mathcal{T}(H)$ is the dual space of $\mathcal{B}_0(H)$, here $\mathcal{B}_0(H)$ denotes the Banach space of all compact operators on H. It follows that, $\rho_{A(k)}$ converges to ρ_A under the weak star topology $\sigma(\mathcal{T}(H), \mathcal{B}_0(H))$. It is known from [29] that the weak-star topology coincides with the trace norm topology on $\mathcal{S}(H)$. Therefore, we conclude that $\rho_{A(k)} \to \rho_A$ as $k \to \infty$ under the trace norm topology.

Similarly, one can show that $\rho_k \to \rho$ as $k \to \infty$ implies that $\rho_{B(k)} \to \rho_B$ as $k \to \infty$. □

This proposition also implies that the trace operation is completely bounded under the trace norm topology on the set of all states.

2 Detecting entanglement by inequalities induced from the CCNR criterion

The main result of this section is the following.

Theorem 1. Let H_A and H_B be complex separable Hilbert spaces with $\dim H_A \otimes H_B = +\infty$, $\rho \in \mathcal{S}_{\mathrm{sep}}(H_A \otimes H_B)$. Then

$$\|(\rho - \rho_A \otimes \rho_B)^R\|_{\mathrm{Tr}} \leqslant \sqrt{[1 - \mathrm{Tr}(\rho_A^2)][1 - \mathrm{Tr}(\rho_B^2)]} \tag{6}$$

and

$$\|(\rho - \rho_A \otimes \rho_B)^{T_B}\|_{\mathrm{Tr}} \leqslant 2\sqrt{[1 - \mathrm{Tr}(\rho_A^2)][1 - \mathrm{Tr}(\rho_B^2)]}, \tag{7}$$

where $\rho_A = \mathrm{Tr}_B(\rho)$, $\rho_B = \mathrm{Tr}_A(\rho)$, and ρ^R stands for the realignment operator of ρ.

There are three equivalent definitions of the realignment operator of an operator in $C_2(H_A \otimes H_B)$ [26], one of them is the following:

Lemma 1 (Guo et al. [26]). Let H_A and H_B be complex Hilbert spaces with $\dim H_A \otimes H_B = +\infty$ and let $C \in C_2(H_A \otimes H_B)$ be a Hilbert-Schmidt operator with $C = \sum_k A_k \otimes B_k$, where $A_k = \sum_{m,n} a_{mn}^{(k)} |m\rangle\langle n| \in C_2(H_A)$, $B_k = \sum_{\mu,\nu} b_{\mu\nu}^{(k)} |\mu\rangle\langle\nu| \in C_2(H_B)$ and the series converges in Hilbert-Schmidt norm. Then

$$C^R = \sum_k |A_k\rangle\langle B_k|, \tag{8}$$

where the series converges in Hilbert-Schmidt norm, $|A_k\rangle = \sum_{m,n} a_{mn}^{(k)} |m\rangle|n\rangle$, $|B_k\rangle = \sum_{\mu,\nu} b_{\mu\nu}^{(k)} |\mu\rangle|\nu\rangle$, $\langle B_k|$ denotes the transposition of $|B_k\rangle$.

In order to prove Theorem 1, some more lemmas are needed. The following lemma is well known for mathematicians and we include a proof of it here for readers' convenience.

Lemma 2. Let H_A and H_B be complex separable Hilbert spaces with $\dim H_A \otimes H_B = +\infty$, $A \in C_p(H_A)$, $B \in C_p(H_B)$ and $1 \leqslant p < +\infty$. Then $A \otimes B \in C_p(H_A \otimes H_B)$, and further more,

$$\|A \otimes B\|_p = \|A\|_p \|B\|_p.$$

Proof. Let $A = U_1 D_1 V_1$ and $B = U_2 D_2 V_2$ be the singular value decomposition of A and B, respectively, where $D_1 = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots)$ and $D_2 = \mathrm{diag}(\lambda_1', \lambda_2', \cdots, \lambda_n', \cdots)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant \cdots$ and $\lambda_1' \geqslant \lambda_2' \geqslant \cdots \geqslant \lambda_n' \geqslant \cdots$. It follows that

$$\|A\|_p = \Big(\sum_i \lambda_i^p\Big)^{\frac{1}{p}} \quad \text{and} \quad \|B\|_p = \Big(\sum_i \lambda_i'^p\Big)^{\frac{1}{p}}.$$

Write $U_1 \otimes U_2 = U$, $D_1 \otimes D_2 = D$ and $V_1 \otimes V_2 = V$. Then we have $A \otimes B = (U_1 D_1 V_1) \otimes (U_2 D_2 V_2) = (U_1 \otimes U_2)(D_1 \otimes D_2)(V_1 \otimes V_2) = UDV$. Since D is a diagonal operator with diagonal entries $\{\lambda_i \lambda_j'\}$, one sees that

$$\begin{aligned}\|A \otimes B\|_p &= \Big(\sum_{i,j} \lambda_i^p \lambda_j'^p\Big)^{\frac{1}{p}} = \Big[\sum_i \lambda_i^p \Big(\sum_j \lambda_j'^p\Big)\Big]^{\frac{1}{p}} \\ &= \Big(\sum_i \lambda_i^p\Big)^{\frac{1}{p}} \Big(\sum_j \lambda_j'^p\Big)^{\frac{1}{p}} = \|A\|_p \|B\|_p,\end{aligned}$$

as desired. □

Lemma 3. Let H_A and H_B be complex separable Hilbert spaces with $\dim H_A \otimes H_B = +\infty$ and $\{\rho_k\}$ be a sequence in $\mathcal{S}_{\mathrm{sep}}(H_A \otimes H_B)$. Then $\{\rho_k\}$ converges to ρ in trace norm implies

$$\lim_{k \to \infty} \rho_k^{T_B} = \rho^{T_B} \tag{9}$$

in trace norm.

Proof. Since ρ_k converges to ρ in trace norm implies ρ_k converges to ρ entry-wise, thus $\rho_k^{T_B}$ converges to ρ^{T_B} entry-wise as well. And it is obvious that ρ is separable, thus ρ^{T_B} is also a state. This implies that $\lim_{k \to \infty} \rho_k^{T_B} = \rho^{T_B}$ with respect to the trace norm since the trace norm topology coincides with the weak-star topology on $\mathcal{S}(H_A \otimes H_B)$. □

Now we are ready to give our proof of the main result.

The proof of Theorem 1. We prove the inequality (6) first. Denote by $\mathcal{S}_{s-p}$ the set of all separable pure states in $\mathcal{S}(H_A \otimes H_B)$. If ρ is separable, then it admits a representation of the Bochner integral [21]

$$\rho = \int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B) d\mu(\rho^A \otimes \rho^B), \tag{10}$$

where μ is a Borel probability measure on $\mathcal{S}_{s-p}$, $\rho^A \otimes \rho^B \in \mathcal{S}_{s-p}$ and $\varphi : \mathcal{S}_{s-p} \to \mathcal{S}_{s-p}$ is a measurable function. It immediately follows from proposition 1 that

$$\rho_A = \int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B)^A d\mu(\rho^A \otimes \rho^B), \tag{11}$$

and

$$\rho_B = \int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B)^B d\mu(\rho^A \otimes \rho^B), \tag{12}$$

where $\varphi(\rho^A \otimes \rho^B)^A = \mathrm{Tr}_B[\varphi(\rho^A \otimes \rho^B)]$, $\varphi(\rho^A \otimes \rho^B)^B = \mathrm{Tr}_A[\varphi(\rho^A \otimes \rho^B)]$.

Observe that $\sigma^A \otimes \sigma^B \in \mathcal{S}_{s-p}$ and

$$\begin{aligned}
&\rho - \rho_A \otimes \rho_B \\
&= \int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B)^A \otimes \varphi(\rho^A \otimes \rho^B)^B d\mu(\rho^A \otimes \rho^B) \\
&\quad - \Big(\int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B)^A d\mu(\rho^A \otimes \rho^B) \Big) \\
&\quad \otimes \Big(\int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B)^B d\mu(\rho^A \otimes \rho^B) \Big) \\
&= \int_{\mathcal{S}_{s-p}} \Big(\int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B)^A d\mu(\sigma^A \otimes \sigma^B) \Big) \\
&\quad \otimes \varphi(\rho^A \otimes \rho^B)^B d\mu(\rho^A \otimes \rho^B) \\
&\quad - \Big(\int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B)^A d\mu(\rho^A \otimes \rho^B) \Big) \\
&\quad \otimes \Big(\int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B)^B d\mu(\rho^A \otimes \rho^B) \Big) \\
&= \int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B)^A \otimes \varphi(\rho^A \otimes \rho^B)^B \\
&\quad \cdot d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B) \\
&\quad - \int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} \varphi(\sigma^A \otimes \sigma^B)^A \\
&\quad \otimes \varphi(\rho^A \otimes \rho^B)^B d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B) \\
&= \int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} (\varphi(\rho^A \otimes \rho^B)^A \otimes \varphi(\rho^A \otimes \rho^B)^B \\
&\quad - \varphi(\sigma^A \otimes \sigma^B)^A \otimes \varphi(\rho^A \otimes \rho^B)^B) \\
&\quad \cdot d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B) \\
&= \int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} (\varphi(\rho^A \otimes \rho^B)^A - \varphi(\sigma^A \otimes \sigma^B)^A) \\
&\quad \otimes \varphi(\rho^A \otimes \rho^B)^B d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B) \\
&= \frac{1}{2} \int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} (\varphi(\rho^A \otimes \rho^B)^A - \varphi(\sigma^A \otimes \sigma^B)^A) \\
&\quad \otimes (\varphi(\rho^A \otimes \rho^B)^B - \varphi(\sigma^A \otimes \sigma^B)^B) \\
&\quad \cdot d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B).
\end{aligned}$$

Thus we arrive at

$$\begin{aligned}
&(\rho - \rho_A \otimes \rho_B)^R \\
&= \frac{1}{2} \int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} [(\varphi(\rho^A \otimes \rho^B)^A - \varphi(\sigma^A \otimes \sigma^B)^A) \\
&\quad \otimes (\varphi(\rho^A \otimes \rho^B)^B - \varphi(\sigma^A \otimes \sigma^B)^B)]^R \\
&\quad \cdot d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B)
\end{aligned}$$

with respect to the Hilbert-Schmidt norm since the realignment operation is continuous in the Hilbert-Schmidt norm [26]. It turns out that

$$\begin{aligned}
&\|(\rho - \rho_A \otimes \rho_B)^R\|_{\mathrm{Tr}} \\
&\leqslant \frac{1}{2} \int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} \|[(\varphi(\rho^A \otimes \rho^B)^A - \varphi(\sigma^A \otimes \sigma^B)^A) \\
&\quad \otimes (\varphi(\rho^A \otimes \rho^B)^B - \varphi(\sigma^A \otimes \sigma^B)^B)]^R\|_{\mathrm{Tr}} \\
&\quad \cdot d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B).
\end{aligned}$$

On the other hand, we let $\varphi(\sigma^A \otimes \sigma^B)^A = |x\rangle\langle x|$, $\varphi(\sigma^A \otimes \sigma^B)^A = |y\rangle\langle y|$, $\varphi(\rho^A \otimes \rho^B)^B = |f\rangle\langle f|$ and $\varphi(\sigma^A \otimes \sigma^B)^B = |g\rangle\langle g|$, where $|x\rangle = (x_1, x_2, \cdots, x_n, \cdots)^T$, $|y\rangle = (y_1, y_2, \cdots, y_n, \cdots)^T \in H_A$, $|f\rangle = (f_1, f_2, \cdots, f_n, \cdots)^T$ and $|g\rangle = (g_1, g_2, \cdots, g_n, \cdots)^T \in H_B$. Then

$$\begin{aligned}
&\|(\varphi(\sigma^A \otimes \sigma^B)^A - \varphi(\sigma^A \otimes \sigma^B)^A) \otimes (\varphi(\rho^A \otimes \rho^B)^B \\
&\quad - \varphi(\sigma^A \otimes \sigma^B)^B)^R\|_{\mathrm{Tr}} \\
&= \|(|\varphi(\sigma^A \otimes \sigma^B)^A\rangle - |\varphi(\sigma^A \otimes \sigma^B)^A\rangle) \cdot (\langle\varphi(\rho^A \otimes \rho^B)^B| \\
&\quad - \langle\varphi(\sigma^A \otimes \sigma^B)^B|)\|_{\mathrm{Tr}} \\
&= \Big[\sum_{i,j} (x_i \bar{x}_j - y_i \bar{y}_j)(\bar{x}_i x_j - \bar{y}_i y_j) \Big]^{\frac{1}{2}} \\
&\quad \cdot \Big[\sum_{i,j} (f_i \bar{f}_j - g_i \bar{g}_j)(\bar{f}_i f_j - \bar{g}_i g_j) \Big]^{\frac{1}{2}} \\
&= \Big[\sum_{i,j} (|x_i x_j|^2 + |y_i y_j|^2 - x_i \bar{x}_j \bar{y}_i y_j - \bar{x}_i x_j y_i \bar{y}_j) \Big]^{\frac{1}{2}} \\
&\quad \cdot \Big[\sum_{i,j} (|f_i f_j|^2 + |g_i g_j|^2 - f_i \bar{f}_j \bar{g}_i g_j - \bar{f}_i f_j g_i \bar{g}_j) \Big]^{\frac{1}{2}} \\
&= \Big(2 - \sum_{i,j} (x_i \bar{x}_j \bar{y}_i y_j + \bar{x}_i x_j y_i \bar{y}_j)\Big)^{\frac{1}{2}} \\
&\quad \cdot (2 - \sum_{i,j} (f_i \bar{f}_j \bar{g}_i g_j + \bar{f}_i f_j g_i \bar{g}_j))^{\frac{1}{2}} \\
&= 2[(1 - \mathrm{Tr}(\varphi(\sigma^A \otimes \sigma^B)^A \varphi(\sigma^A \otimes \sigma^B)^A)) \\
&\quad \cdot (1 - \mathrm{Tr}(\varphi(\rho^A \otimes \rho^B)^B \varphi(\sigma^A \otimes \sigma^B)^B))]^{\frac{1}{2}}.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\|(\rho - \rho_A \otimes \rho_B)^R\|_{\mathrm{Tr}} \\
&\leqslant \int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} [1 - \mathrm{Tr}(\varphi(\sigma^A \otimes \sigma^B)^A \varphi(\sigma^A \otimes \sigma^B)^A)]^{\frac{1}{2}} \\
&\quad \cdot [1 - \mathrm{Tr}(\varphi(\rho^A \otimes \rho^B)^B \varphi(\sigma^A \otimes \sigma^B)^B)]^{\frac{1}{2}} \\
&\quad \cdot d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B) \\
&\leqslant \Big[\int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} \|(1 - \mathrm{Tr}(\varphi(\sigma^A \otimes \sigma^B)^A \varphi(\sigma^A \otimes \sigma^B)^A)) \\
&\quad \cdot d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B) \Big]^{\frac{1}{2}} \\
&\quad \cdot \Big[\int_{\mathcal{S}_{s-p}} \int_{\mathcal{S}_{s-p}} \|(1 - \mathrm{Tr}(\varphi(\sigma^A \otimes \sigma^B)^B \varphi(\sigma^A \otimes \sigma^B)^B)) \\
&\quad \cdot d\mu(\rho^A \otimes \rho^B) d\mu(\sigma^A \otimes \sigma^B) \Big]^{\frac{1}{2}} \\
&= [(1 - \mathrm{Tr}(\rho_A^2))(1 - \mathrm{Tr}(\rho_B^2))]^{\frac{1}{2}}.
\end{aligned}$$

For the second inequality above the Cauchy-Schwarz inequality is used.

Now we begin to show the inequality (7). If ρ is countably separable with $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$, then, by Lemma 2, we have

$$\begin{aligned} &\|(\rho - \rho_A \otimes \rho_B)^{T_B}\|_{\rm Tr} \\ &= \|\frac{1}{2}\sum_{i,j} p_i p_j (\rho_i^A - \rho_j^A) \otimes (\overline{\rho_i^B} - \overline{\rho_j^B})\|_{\rm Tr} \\ &\leqslant \frac{1}{2}\sum_{i,j} p_i p_j \|(\rho_i^A - \rho_j^A) \otimes (\overline{\rho_i^B} - \overline{\rho_j^B})\|_{\rm Tr} \\ &= \frac{1}{2}\sum_{i,j} p_i p_j \|\rho_i^A - \rho_j^A\|_{\rm Tr} \|\overline{\rho_i^B} - \overline{\rho_j^B}\|_{\rm Tr} \\ &= \frac{1}{2}\sum_{i,j} p_i p_j \|\rho_i^A - \rho_j^A\|_{\rm Tr} \|\rho_i^B - \rho_j^B\|_{\rm Tr} \end{aligned}$$

since

$$\begin{aligned} &\rho - \rho_A \otimes \rho_B \\ &= \sum_i p_i \rho_i^A \otimes \rho_i^B - (\sum_i p_i \rho_i^A) \otimes (\sum_j p_j \rho_j^B) \\ &= \sum_{i,j} (p_j \rho_i^A) \otimes (p_i \rho_i^B) - (\sum_i p_i \rho_i^A) \otimes (\sum_j p_j \rho_j^B) \\ &= \sum_{i,j} [(p_j \rho_i^A) \otimes (p_i \rho_i^B) - (p_i \rho_i^A) \otimes (p_j \rho_j^B)] \\ &= \sum_{i,j} p_i p_j (\rho_i^A \otimes \rho_i^B - \rho_i^A \otimes \rho_j^B) \\ &= \frac{1}{2}\sum_{i,j} p_i p_j (\rho_i^A - \rho_j^A) \otimes (\rho_i^B - \rho_j^B). \end{aligned}$$

Noticing that ${\rm rank}(\rho_i^A - \rho_j^A) \leqslant 2$, ${\rm Tr}(\rho_i^A - \rho_j^A) = 0$ and $(\rho_i^A - \rho_j^A)^\dagger = \rho_i^A - \rho_j^A$, we can conclude that the eigenvalues of $\rho_i^A - \rho_j^A$ are $\alpha, -\alpha, \alpha \geqslant 0$, which implies that the singular values of $\rho_i^A - \rho_j^A$ are α, α. It follows from ${\rm Tr}[(\rho_i^A - \rho_j^A)^2] = 2\alpha^2$ that $\|\rho_i^A - \rho_j^A\|_{\rm Tr} = \sqrt{2{\rm Tr}[(\rho_i^A - \rho_j^A)^2]} = 2\sqrt{1 - {\rm Tr}(\rho_i^A \rho_j^A)}$. Similarly, we have $\|\rho_i^B - \rho_j^B\|_{\rm Tr} = 2\sqrt{1 - {\rm Tr}(\rho_i^B \rho_j^B)}$. Thus, by Cauchy-Schwarz inequality, we arrive at

$$\|(\rho - \rho_A \otimes \rho_B)^{T_B}\|_{\rm Tr} \leqslant 2\sqrt{[1 - {\rm Tr}(\rho_A^2)][1 - {\rm Tr}(\rho_B^2)]}.$$

If ρ is not countably separable, then there exists a sequence of countably separable states $\{\sigma_n\}$ such that $\lim_{n\to\infty} \sigma_n = \rho$ with respect to the trace norm. It follows from Proposition 1 and Lemma 3 that,

$$\begin{aligned} &\|(\rho - \rho_A \otimes \rho_B)^{T_B}\|_{\rm Tr} \\ &= \lim_{n\to\infty} \|(\sigma_n - \sigma_{A(n)} \otimes \sigma_{B(n)})^{T_B}\|_{\rm Tr} \\ &\leqslant \lim_{n\to\infty} 2\sqrt{[1 - {\rm Tr}(\sigma_{A(n)}^2)][1 - {\rm Tr}(\sigma_{B(n)}^2)]} \\ &= \sqrt{[1 - {\rm Tr}(\rho_A^2)][1 - {\rm Tr}(\rho_B^2)]}, \end{aligned}$$

where $\sigma_{A(n)} = {\rm Tr}_B(\sigma_n)$ and $\sigma_{B(n)} = {\rm Tr}_A(\sigma_n)$. □

We assert that inequality (6) can detect all states that can be recognized by the CCNR criterion. In fact, if $\|\rho^R\|_{\rm Tr} > 1$, then $\|(\rho - \rho_A \otimes \rho_B)^R\|_{\rm Tr} \geqslant \|\rho^R\|_{\rm Tr} - \|(\rho_A \otimes \rho_B)^R\|_{\rm Tr} = \|\rho^R\|_{\rm Tr} - \||\rho_A\rangle\langle\rho_B|\|_{\rm Tr} = \|\rho^R\|_{\rm Tr} - \|\rho_A\|_2 \cdot \|\rho_B\|_2 > 1 - \|\rho_A\|_2 \cdot \|\rho_B\|_2 \geqslant \sqrt{[1 - {\rm Tr}(\rho_A^2)][1 - {\rm Tr}(\rho_B^2)]}$. In what follows, we will show that the inequality (6) in Theorem 1 provides a criterion that can detect some PPT entangled state ρ with $\|\rho^R\|_{\rm Tr} \leqslant 1$.

Example. Let H_A and H_B be complex Hilbert spaces with orthonormal bases $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$ and $\{|0'\rangle, |1'\rangle, |2'\rangle, \ldots\}$, respectively. Let $0 < a < 1$ and

$$\begin{aligned} \tilde{\rho} = &\frac{a}{8a+1}(|00'\rangle\langle 00'| + |01'\rangle\langle 01'| + |02'\rangle\langle 02'| \\ &+ |00'\rangle\langle 11'| + |00'\rangle\langle 22'| + |11'\rangle\langle 00'| + |22'\rangle\langle 00'| \\ &+ |10'\rangle\langle 10'| + |11'\rangle\langle 11'| + |12'\rangle\langle 12'| \\ &+ |11'\rangle\langle 22'| + |22'\rangle\langle 11'| + |21'\rangle\langle 21'|) \\ &+ \frac{1+a}{2}(|20'\rangle\langle 20'| + |22'\rangle\langle 22'|) \\ &+ \frac{\sqrt{1-a^2}}{2}(|20'\rangle\langle 22'| + |22'\rangle\langle 20'|). \end{aligned}$$

Write

$$\tilde{\rho_\epsilon} = \epsilon\tilde{\rho} + (1-\epsilon)\frac{P_3}{9}, \quad P_3 = \sum_{i,j=0}^{2} |i\rangle\langle i| \otimes |j'\rangle\langle j'|.$$

If $\dim H_A = \dim H_B = 3$, it is obvious that

$$\tilde{\rho} = \hat{a}\left(\begin{array}{ccc|ccc|ccc} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1+a) & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1}{2}(1+a) \end{array}\right)$$

(is a bound entangled state [30]) and

$$\tilde{\rho_\epsilon} = \epsilon\rho + (1-\epsilon)\frac{I}{9}, \quad \hat{a} = \frac{1}{8a+1}.$$

It is showed in [24] that, for $3 \otimes 3$ system, $\tilde{\rho_\epsilon}$ is entangled when $\epsilon \geqslant 0.9955$ and $a = 0.236$ applying the CCNR criterion. Using inequality (2), it is found that $\tilde{\rho_\epsilon}$ is still entangled when $\epsilon = 0.9939$ and $a = 0.232$ [25]. It is straightforward that

$$\tilde{\rho_\epsilon} \text{ is entangled whenever } \epsilon \geqslant 0.9939 \text{ and } a = 0.232.$$

Define

$$\sigma = \sum_{i=3}^{+\infty} p_i |i\rangle\langle i| \otimes |i'\rangle\langle i'|, \quad p_i \geqslant 0, \ \sum_{i=3}^{+\infty} p_i = 1.$$

It is clear that σ is separable. Now we let

$$\tilde{\rho_{\epsilon,c}} = c\tilde{\rho_\epsilon} + (1-c)\sigma, \quad 0 \leqslant c \leqslant 1, \tag{13}$$

then $\|\tilde{\rho_{\epsilon,c}}^R\|_{\rm Tr} = c\|\tilde{\rho_\epsilon}^R\|_{\rm Tr} + 1 - c$ since $\|\sigma^R\|_{\rm Tr} = 1$ and it is evident that $\tilde{\rho_{\epsilon,c}}^{T_{A\backslash B}} \geqslant 0$. On the other hand, one can find that $\check{\rho_A} = {\rm Tr}_A(\tilde{\rho_{\epsilon,c}}) = c{\rm Tr}_A(\tilde{\rho_\epsilon}) + (1-c){\rm Tr}_A(\sigma)$ and $\check{\rho_B} = {\rm Tr}_B(\tilde{\rho_{\epsilon,c}}) = c{\rm Tr}_B(\tilde{\rho_\epsilon}) + (1-c){\rm Tr}_B(\sigma)$. Together with the fact that the trace operation is completely bounded, we can conclude that there exists some $0 < c_0 < 1$, $0.9939 \leqslant \epsilon_0 < 0.9955$ and $0 < \varepsilon < 0.232$ such that $\tilde{\rho_{\epsilon,c}}$ violates the inequality (6) whenever $c > c_0$, $0.9939 \leqslant \epsilon < \epsilon_0$ and $0.232 - \varepsilon < a < 0.232 + \varepsilon$ while $\|\tilde{\rho_{\epsilon,c}}^R\|_{\rm Tr} \leqslant 1$ and $\tilde{\rho_{\epsilon,c}}^{T_{A\backslash B}} \geqslant 0$ whenever $c > c_0$, $0.9939 \leqslant \epsilon < \epsilon_0$ and $0.232 - \varepsilon < a < 0.232 + \varepsilon$.

3 Conclusions

In conclusion, an entanglement criterion beyond the CCNR criterion for infinite-dimensional systems is proposed: Based on the CCNR criterion for infinite-dimensional systems, we highlighted the relation between separable states and the reduced states via realignment operation or partial transposition. It is showed that the obtained inequality can detect more entangled states than the CCNR criterion. It should be stressed here that the proof of our main result needs new tools which is very different from the finite-dimensional case.

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