Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physics Letters B 632 (2006) 417–421
PHYSICS LETTERS B
www.elsevier.com/locate/physletb

Spinors in the hyperbolic algebra

S. Ulrych

Wehrenbachhalde 35, CH-8053 Zürich, Switzerland

Received 10 October 2005; received in revised form 17 October 2005; accepted 19 October 2005
Available online 26 October 2005
Editor: N. Glover

Abstract

The three-dimensional universal complex Clifford algebra $\bar{\boldsymbol{C}}_{3,0}$ is used to represent relativistic vectors in terms of paravectors. In analogy to the Hestenes spacetime approach spinors are introduced in an algebraic form. This removes the dependance on an explicit matrix representation of the algebra.

PACS: 03.65.Fd; 02.10.De; 11.30.Cp; 02.10.Hh; 02.20.Qs

MSC: 81R25; 15A66; 30G35; 15A33; 20H25

Keywords: Hyperbolic complex Clifford algebra; Algebraic spinor; Hyperbolic numbers; Paracomplex numbers; Split-complex numbers

1. Introduction

Over the last years there is growing interest in the Clifford algebra approach to spacetime that has been initiated by Hestenes [1–4]. The algebraic representation of geometry has in general advantages compared to the conventional description in terms of column vectors and matrices. An overview of applications is given by Doran and Lasenby [5] and by Gull et al. [6].

Beside the approach to relativistic physics in terms of the $\boldsymbol{R}_{1,3}$ Dirac algebra, there is the less noticed approach of Baylis which is based on $\boldsymbol{R}_{3,0}$ paravectors [7]. Baylis derived a new representation of electrodynamics [8] with these paravectors. This algebra has been introduced originally by Sobczyk in the spacetime vector analysis [9]. In a recent work [10] this algebra is generalized to the universal complex Clifford algebra $\bar{\boldsymbol{C}}_{3,0}$. The structural difference compared to Baylis appears in the shape of the hyperbolic unit, which plays an integral part in the complex formalism.

For more details on the hyperbolic numbers and their properties it is referred to the references in [10]. However, it should be mentioned that hyperbolic numbers are used also within general relativity, where the hyperbolic numbers are also denoted as paracomplex or split-complex numbers. Paracomplex projective models and harmonic maps were investigated by Erdem [11–13]. A survey on paracomplex geometry, para-Hermitian, and para-Kaehler manifolds has been given by Cruceanu et al. [14,15]. Solutions of Minkowskian sigma models generated by hyperbolic numbers were considered by Lambert et al. [16,17]. Zhong generated new solutions of the stationary axisymmetric Einstein equations with hyperbolic numbers [18]. He investigated hyperbolic complex linear symmetry groups and their local gauge transformation actions [19]. Furthermore, the hyperbolic complexification of Hopf algebras [20]. Moffat [21] has interpretated the hyperbolic number as fermion number. This interpretation has led to fundamental explanation of stability of fermionic matter.

As a supplement to the references in [10] it is mentioned that introductions to hyperbolic numbers including further references are given by Sobczyk [22], Borota and Osler [23]. Hyperbolic numbers are applied to integrable systems by Bracken and Hayes [24,25]. Linear and quasilinear complex equations are investigated by Wen based on hyperbolic numbers [26]. A slightly different structure than the algebra used in this work have the so-called paraquaternions (or split-quaternions [27]) used, e.g., by Blažić [28].

E-mail address: stefan.ulrych@bluewin.ch (S. Ulrych).

0370-2693 © 2005 Elsevier B.V. Open access under CC BY license.
doi:10.1016/j.physletb.2005.10.040

The hyperbolic numbers form, together with complex numbers and quaternions, the fundamental building blocks in the classification of Clifford algebras. Porteous [29] gives an overview of real and complex Clifford algebras for geometries with arbitrary signatures. Beside the real Clifford algebras there exist four different types of complex algebras. Porteous derives explicit matrix representations for all algebras. Spinors are elements of a minimal left ideal, which can be represented as column vectors corresponding to the matrix representation of the algebra. Such a representation has been used in [10].

Though this conventional picture is familiar to physicists, Hestenes actually promoted the full algebraic representation of vectors and spinors. The theory should be free of any explicit matrix representations. Hestenes achieved this for non-relativistic and relativistic physics. It is the intention of this work to apply these ideas explicitly to the three-dimensional complex paravector algebra.

2. Hyperbolic algebra

Vector spaces can be defined over the commutative ring of hyperbolic numbers $z \in \boldsymbol{H}$

$$z = x + iy + jv + ijw, \quad x, y, v, w \in \boldsymbol{R}, \tag{1}$$

where the hyperbolic unit j has the property $j^2 = 1$. In the terminology of Clifford algebras the hyperbolic numbers defined in this way are represented by $\bar{\boldsymbol{C}}_{1,0}$, i.e., they correspond to the universal one-dimensional complex Clifford algebra (see Porteous [29]).

Beside the grade involution, two anti-involutions play a major role in the description of Clifford algebras and their structure, conjugation and reversion. Conjugation changes the sign of the complex and the hyperbolic unit

$$\bar{z} = x - iy - jv + ijw. \tag{2}$$

Reversion, denoted as $z^\dagger$, changes only the sign of the complex unit. Anti-involutions reverse the ordering in the multiplication, e.g., $(ab)^\dagger = b^\dagger a^\dagger$. This becomes important when non-commuting elements of an algebra are considered. With respect to conjugation the square of the hyperbolic number,

$$|z|^2 = z\bar{z}, \tag{3}$$

can be calculated as

$$z\bar{z} = x^2 + y^2 - v^2 - w^2 + 2ij(xw - yv). \tag{4}$$

The hyperbolic numbers form the basis of the hyperbolic paravector algebra. A Minkowski vector $x^\mu = (x^0, x^i) \in \boldsymbol{R}^{3,1}$ is represented in terms of the hyperbolic algebra as

$$x = x^\mu e_\mu. \tag{5}$$

The basis elements $e_\mu = (e_0, e_i)$ include the unity and the Pauli algebra multiplied by the hyperbolic unit j

$$e_\mu = (1, j\sigma_i). \tag{6}$$

The algebra can be complexified with either the hyperbolic or the complex unit. The full structure is equivalent to the universal three-dimensional complex Clifford algebra $\bar{\boldsymbol{C}}_{3,0}$.

The scalar product of two vectors is defined as

$$x \cdot y = \frac{1}{2}(x\bar{y} + y\bar{x}) = \langle x\bar{y}\rangle_+. \tag{7}$$

The wedge product is given as

$$x \wedge y = \frac{1}{2}(x\bar{y} - y\bar{x}) = \langle x\bar{y}\rangle_-. \tag{8}$$

The wedge product corresponds to a so-called biparavector, which can be used for the description of the electromagnetic field or the relativistic angular momentum (see also Baylis [8]). A special notation has been introduced indicating the symmetric and anti-symmetric contributions of a geometric product

$$x\bar{y} = \langle x\bar{y}\rangle_+ + \langle x\bar{y}\rangle_-. \tag{9}$$

The basis elements of the $\bar{\boldsymbol{C}}_{3,0}$ paravector algebra can be considered as the basis vectors of the relativistic vector space. These basis elements form a non-Cartesian orthogonal basis with respect to the scalar product defined in Eq. (7)

$$e_\mu \cdot e_\nu = g_{\mu\nu}, \tag{10}$$

where $g_{\mu\nu}$ is the metric tensor of the Minkowski space.

The group $SU(2, \boldsymbol{H})$ corresponds to the spin group of the hyperbolic algebra and its elements can be used to express rotations and boosts of the paravectors. The rotation of a paravector can be expressed as

$$x \to x' = RxR^\dagger. \tag{11}$$

For the boosts one finds the transformation rule

$$x \to x' = BxB^\dagger. \tag{12}$$

Rotations and boosts are given as

$$R = \exp\bigl(-i\theta^i\sigma_i/2\bigr), \qquad B = \exp\bigl(j\xi^i\sigma_i/2\bigr). \tag{13}$$

The infinitesimal generators of a Lorentz transformation can be identified as

$$J_i = \sigma_i/2, \qquad K_i = ij\sigma_i/2. \tag{14}$$

The generators satisfy the Lie algebra of the Lorentz group.

Boosts are invariant under reversion $B^\dagger = B$, whereas the conjugated boost corresponds to the inverse $\bar{B} = B^{-1}$. For rotations reversion and conjugation correspond both to the inverse $R^\dagger = \bar{R} = R^{-1}$. The effect of conjugation, reversion, and graduation on the used hypercomplex units is displayed in Table 1. Note, that graduation is an involution, which does not reverse the ordering in a product, i.e., $\widehat{ab} = \hat{a}\hat{b}$. Conjugation, reversion, and graduation are related by $\bar{a} = \hat{a}^\dagger$.

This was a brief summary of the most important facts. A more detailed representation of the hyperbolic algebra can be found in [10].

Table 1
Effect of conjugation, reversion, and graduation on the used hypercomplex units

a	$\bar{a}$	$a^\dagger$	$\hat{a}$
e_0	+	+	+
e_i	−	+	−
σ_i	+	+	+
i	−	−	+
j	−	+	−

3. Spinors in the hyperbolic algebra

The relationship between relativistic spinors and vectors can be derived in the same way as in non-relativistic physics. Starting from a parameterization of a normalized Minkowski vector in terms of spherical coordinates, which can be generated from a standard vector by a vector transformation, the equivalent object in spinor space is derived from the corresponding spin transformation. A normalized spacelike vector x^μ can be represented as

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \sinh\xi \\ \cosh\xi\sin\theta\cos\phi \\ \cosh\xi\sin\theta\sin\phi \\ \cosh\xi\cos\theta \end{pmatrix}. \tag{15}$$

In the limit of $\xi \to 0$ the vector reduces to a non-relativistic vector in spherical coordinates. The vector can be obtained from a standard vector $x^\mu = (0,0,0,1)$ with a Lorentz transformation of the form

$$L = e^{-i\phi J_3} e^{-i\theta J_2} e^{-i\xi K_3}. \tag{16}$$

With the generators of Eq. (14) the corresponding spin transformation can be written as

$$S = e^{-i\phi\sigma_3/2} e^{-i\theta\sigma_2/2} e^{j\xi\sigma_3/2}. \tag{17}$$

The spinor, corresponding to the above vector, is obtained in the conventional picture from a multiplication of the two-component standard spinor $\chi^i = (1,0)$ by the above spin transformation

$$\psi^i = S\chi^i. \tag{18}$$

The elements of the Pauli algebra are represented here as 2×2 matrices. The hyperbolic spinor has two components $\psi^i \in \bar{\boldsymbol{H}}^2$. The bar symbol indicates that the correlation, which maps the elements of the spinor to its dual space, is defined with conjugation as given in Eq. (2). With the relation $\bar{S} = S^{-1}$ it is easy to show that the spinor is normalized

$$\bar{\psi}_i\psi^i = 1. \tag{19}$$

This representation provides a consistent framework for relativistic calculations [10]. However, this picture requires an explicit matrix representation of the algebra. Hestenes [1] suggested to identify the spinor directly with the spinor transformation itself to obtain a spinor in a pure algebraic form. This concept can be adopted also in the current context. The Clifford algebraic spinor is therefore defined as

$$\psi = S. \tag{20}$$

The spinor can be expanded into a component structure with an even number of basis vectors

$$\psi = \psi^0 + \frac{\psi^{\mu\nu}}{2!}\langle e_\mu \bar{e}_\nu\rangle_- + \frac{\psi^{\mu\nu\sigma\rho}}{4!}\langle e_\mu\bar{e}_\nu e_\sigma\bar{e}_\rho\rangle_-, \tag{21}$$

where only the antisymmetric contributions of the algebra products are considered. Note, that this structure is not an element of the even Clifford algebra. It is a mixture of even and odd elements with respect to the grade involution. The reason lies in the paravector algebra and the element $e_0 = 1$, which is invariant under the grade involution. Baylis [8] introduces the terminology of a paravector grade. However, it is shown below that a spinor cannot be identified uniquely as an element of even paravector grade.

The factors in Eq. (21) are introduced by convention. They indicate that not all elements in this expansion are linear independent. The spinor consists of a scalar part, six independent components of a biparavector, and one pseudoscalar contribution. The last term could therefore be expressed also in the simplified form

$$ij\eta = \frac{\psi^{\mu\nu\sigma\rho}}{4!}\langle e_\mu\bar{e}_\nu e_\sigma\bar{e}_\rho\rangle_-, \tag{22}$$

with the pseudoscalar $ij\eta$. The explicit form of the spinor components for the parametrization of Eq. (17) is given in Appendix A. The eight independent components are included also in the two-component spinor of Eq. (18). Explicitly one finds

$$\psi^i = \begin{pmatrix} \psi^0 + i\psi^{21} + j\psi^{30} + ij\psi^{0123} \\ \psi^{31} + i\psi^{32} + j\psi^{10} + ij\psi^{20} \end{pmatrix}. \tag{23}$$

The only non-trivial operators that can be generated by the basis elements of the hyperbolic algebra are ij, $i\sigma_i$, and $j\sigma_i$. Together with the unity they form the subalgebra $\boldsymbol{R}_{3,0}$, which will be denoted here as spinor algebra. The effect of these operators on the spinor has to be investigated to proof the one to one relationship between the spinors given in Eqs. (18) and (20). For the two-component structure the elements of the Pauli algebra have to be replaced by their explicit 2×2 matrix representation. If the algebraic spinor is represented in the form

$$\begin{aligned}\psi = {}& \psi^0 + \psi^{32} i\sigma_1 + \psi^{13} i\sigma_2 + \psi^{21} i\sigma_3 \\ & + \psi^{10} j\sigma_1 + \psi^{20} j\sigma_2 + \psi^{30} j\sigma_3 + ij\psi^{0123},\end{aligned} \tag{24}$$

the proof of this one to one correspondence is straightforward.

From Eq. (24) it is obvious that the spinor can be expanded also into an odd number of basis vectors

$$\psi = \psi^\mu e_\mu + \frac{\psi^{\mu\nu\sigma}}{3!}\langle e_\mu\bar{e}_\nu e_\sigma\rangle_-, \tag{25}$$

which simply leads to a relabelling of the spinor components. This spinor is formed by a paravector and a triparavector. The triparavector is calculated as [8]

$$\begin{aligned}\langle e_\mu\bar{e}_\nu e_\sigma\rangle_- = {}& \frac{1}{3!}(e_\mu\bar{e}_\nu e_\sigma + e_\nu\bar{e}_\sigma e_\mu + e_\sigma\bar{e}_\mu e_\nu \\ & - e_\nu\bar{e}_\mu e_\sigma - e_\mu\bar{e}_\sigma e_\nu - e_\sigma\bar{e}_\nu e_\mu).\end{aligned} \tag{26}$$

The divisor in Eq. (25) indicates again that not all elements of the triparavector are linear independent. In fact, there are only four independent components. One could therefore also write the last term in the simplified form

$$ij\eta^\mu e_\mu = \frac{\psi^{\mu\nu\sigma}}{3!}\langle e_\mu\bar{e}_\nu e_\sigma\rangle_-, \tag{27}$$

where $ij\eta^\mu$ has the structure of a pseudovector. Though this representation looks like a sum of a vector and a pseudovector, keep in mind that in the terminology of Clifford algebras the spinor is considered as an element of a minimal left ideal. It is only multiplied from the left with other elements of the algebra, whereas a vector transforms according to Eqs. (11) and (12).

4. Spinor product

In the conventional picture the scalar product of two spinors is based on the correlation, which maps the elements of the spinor space to their dual space. If the spinor corresponds to a two-component column vector the correlation is represented with transposition and conjugation. One can therefore write

$$\varphi \circ \psi = \bar{\varphi}_i \psi^i, \tag{28}$$

with $\varphi, \psi \in \bar{\boldsymbol{H}}^2$. The Hermitian product will be denoted in the following as spinor product.

Based on Eq. (7) the spinor product of two algebraic spinors can be defined as

$$\varphi \circ \psi = \varphi \cdot \psi + j\varphi \cdot \psi e_3. \tag{29}$$

The second term corresponds to a projection in the direction of the z-axis. An explicit calculation using Eqs. (23) and (24) shows that the above spinor products for the column spinor and the algebraic spinor are equivalent. The square of the spinor product can be calculated in terms of relative coordinates as

$$|\varphi \circ \psi|^2 = \cos^2(\theta/2)(1 + ij \sinh\xi \sin\phi), \tag{30}$$

where ψ has been chosen to be in its standard frame. Note, that the same expression can be derived also in momentum space.

This square appears in physics in the calculation of cross sections, for example in the scattering of polarized electrons by a spinless nucleus like oxygen ^{16}O (see, e.g., Perkins [30]). The spinor ψ can be chosen to represent the spin structure of the incoming electron beam with momentum and polarization ($m_s = +1/2$) aligned in the direction of the z-axis. The scattered electron beam, still polarized in a $m_s = +1/2$ state, corresponds to φ. In the case of elastic scattering Eq. (30) reduces to the factor $\cos^2(\theta/2)$, which is equal to the contribution of the electron spin to the Mott formula in the conventional mathematical formulation of the problem. For inelastic scattering the second term in Eq. (30) appears, which is proportional to the pseudoscalar ij of the hyperbolic algebra.

The square of the spinor product is a factor in the cross section of the process, i.e., the number of electrons counted with a certain direction, energy, and momentum is directly related to Eq. (30). Since this number is clearly a real number, one may ask whether the hyperbolic complex part of Eq. (30) is of physical relevance.

5. Summary

Spinors can be represented in an algebraic form within the three-dimensional complex Clifford algebra $\bar{\boldsymbol{C}}_{3,0}$. The conventional two-component hyperbolic spinor is equivalent to an expansion of even and odd elements of the Clifford algebra. This is in contrast to the common understanding of a spinor as an element of the even Clifford algebra. The reason for this is given by the fact that a paravector algebra is used instead of a vector algebra.

The algebraic spinor must not be multiplied from the right by an element of the Pauli algebra, if the mass operator [10], which can be represented in terms of the spinor algebra $\boldsymbol{R}_{3,0}$, is acting on it. This is required in the $\boldsymbol{R}_{1,3}$ algebra in order to keep the spinor within the even algebra, when it is multiplied by the odd grade Dirac operator.

The scalar product of the Clifford algebra can be used to define a spinor product, which is equivalent to the conventional scalar product of a column spinor.

Appendix A. Algebraic spinors

The components of the Clifford algebraic spinor are listed below. They correspond to the parametrization given in Eq. (17)

$$\begin{aligned}
\psi^0 &= \cos\phi/2 \cos\theta/2 \cosh\xi/2,\\
\psi^{10} &= \cos\phi/2 \sin\theta/2 \sinh\xi/2,\\
\psi^{20} &= \sin\phi/2 \sin\theta/2 \sinh\xi/2,\\
\psi^{30} &= \cos\phi/2 \cos\theta/2 \sinh\xi/2,\\
\psi^{12} &= \sin\phi/2 \cos\theta/2 \cosh\xi/2,\\
\psi^{31} &= \cos\phi/2 \sin\theta/2 \cosh\xi/2,\\
\psi^{32} &= \sin\phi/2 \sin\theta/2 \cosh\xi/2,\\
\psi^{3210} &= \sin\phi/2 \cos\theta/2 \sinh\xi/2.
\end{aligned} \tag{A.1}$$

Note, that the components of the spinor are antisymmetric with respect to their indices. The ordering in the indices has been chosen to give positive values for all components. The elements appear partly with reversed indices in the explicit form of the spinors in Eqs. (23) and (24).

It is an interesting point that in the non-relativistic limit $\xi \to 0$ a vector can be formed that corresponds to a parametrization of a rotation with a 4π symmetry

$$x^i = \begin{pmatrix} \psi^{32} \\ \psi^{13} \\ \psi^{21} \end{pmatrix} = \begin{pmatrix} \sin\phi/2 \sin\theta/2 \\ -\cos\phi/2 \sin\theta/2 \\ -\sin\phi/2 \cos\theta/2 \end{pmatrix}. \tag{A.2}$$

References

[1] D. Hestenes, Space Time Algebra, Gordon and Breach, New York, 1966.
[2] D. Hestenes, G. Sobczyk, Clifford Algebra to Geometric Calculus, Reidel, Dordrecht, 1984.
[3] D. Hestenes, New Foundations for Classical Mechanics, second ed., Kluwer Academic Publishers, Dordrecht, 1999.
[4] D. Hestenes, Am. J. Phys. 71 (2003) 691.
[5] C. Doran, A. Lasenby, Geometric Algebra for Physicists, Cambridge Univ. Press, Cambridge, 2003.
[6] S. Gull, A. Lasenby, C. Doran, Found. Phys. 23 (9) (1993) 1175.
[7] W.E. Baylis, G. Jones, J. Phys. A 22 (1989) 1.
[8] W.E. Baylis, Electrodynamics: A Modern Geometrical Approach, Birkhauser Boston, Cambridge, 1999.
[9] G. Sobczyk, Phys. Lett. A 84 (1981) 45.
[10] S. Ulrych, Phys. Lett. B 625 (2005) 313.
[11] S. Erdem, Math. Proc. Cambridge Philos. Soc. 94 (1983) 483.
[12] S. Erdem, Contrib. Algebra Geom. 38 (1) (1997) 19.
[13] S. Erdem, Contrib. Algebra Geom. 40 (2) (1999) 385.
[14] V. Cruceanu, P.M. Gadea, J.M. Masque, Quad. Istit. Matematico (Messina) 1 (1995) 1.

[15] V. Cruceanu, P. Fortuny, P.M. Gadea, Rocky Mountain J. Math. 26 (1996) 83.
[16] D. Lambert, P. Tombal, Int. J. Theor. Phys. 26 (10) (1987) 943.
[17] D. Lambert, B. Piette, Class. Quantum Grav. 5 (1988) 307.
[18] Z.-Z. Zhong, J. Math. Phys. 26 (10) (1985) 2589.
[19] Z.-Z. Zhong, J. Math. Phys. 26 (3) (1985) 404.
[20] Z.-Z. Zhong, Int. J. Theor. Phys. 38 (2) (1999) 519.
[21] J.W. Moffat, Proceedings of the VIth International School of Gravitation and Cosmology, Erice, 1981, World Scientific, Singapore, 1982.
[22] G. Sobczyk, College Math. J. 26 (1995) 268.
[23] N.A. Borota, T.J. Osler, Math. Comput. Education 36 (2002) 231.
[24] P. Bracken, J. Hayes, Phys. Lett. A 301 (2002) 191.
[25] P. Bracken, J. Hayes, Am. J. Phys. 71 (2003) 726.
[26] G.C. Wen, Linear and Quasilinear Complex Equations of Hyperbolic and Mixed Types, Taylor & Francis, London, 2002.
[27] J. Inoguchi, M. Toda, Acta Appl. Math. 83 (3) (2004) 313.
[28] N. Blažić, Publ. Inst. Math. (Beograd) (N.S.) 60 (74) (1996) 101.
[29] I. Porteous, Clifford Algebras and the Classical Groups, Cambridge Univ. Press, Cambridge, 1995.
[30] D.H. Perkins, Introduction to High Energy Physics, Addison–Wesley, Menlo Park, 1987.