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Integral Equation for Symmetrical Free Vibration of Levy – Plate Having Discontinuous Simple Supports

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Abstract

he principal concerned in this paper is the finite Hankel integral transform method for solving the symmetrical free vibration problem of rectangular thin plate. The considered plate has two opposite simply supported edges and the others are partially simply supported mixed with free. Based on the Levy – plate solution, the mixed boundary conditions between the simple and free supports on the same side of the plate can be written as dual – series equations. However, with the advantages of the present method, these mentioned equations are analytically reduced to homogeneous integral equation of Fredholm – type. Significantly, the nature of moment singularities where the simple support changes to a free edge has been brought into consideration. They can be isolated and treated analytically with a weight function. From this aspect, the attention is placed specifically on deriving an integral equation of the problem considered here instead of the partial differential equation. As a result, those with mixed boundary conditions and the moment singularities in the order of an inverse – square – root type have been successfully obtained.

Keywords: Discontinuous Support, Dual – series equations, Fredholm integral equation, Hankel integral transform, Mixed boundary conditions, Singularities, Vibration

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1. Problem Statement

Within the frameworks of thin plate theory which concerned themselves with mixed boundary value problems, the work of (Nowacki and Olesiak 1956) has considered the vibration, buckling and bending of circular plate having a partially clamped along part of its periphery and simply supported on the remaining part. (Gladwell 1958) has used the complex variable method to analyze the rectangular plate clamped along part of the boundary and either free, or subjected to specified bending moment and shear along the remainder. Then, (Zorski 1958) has considered the bending of a semi – plane or a quarter of the plate with the boundary which mixed between clamped, free and simply supported conditions. The problem was reduced to solve a singular integral equation.

In view of the problems treated by (Nowacki and Olesiak 1956), alternatively (Bartlett 1963) has formulated the vibration and buckling problems through the dual – series equations and then, solved by variation principle to determine the upper and lower bounds for the lowest eigenvalue. Subsequently, (Noble 1965) investigated an approximate solution for the lowest eigenvalue where the dual – series equations were solved by using closed form trigonometric series. (Yang 1968) studied the integral equation solution for a rectangular plate with an internal line support. The problem was formulated by using Green's function and displayed in terms of an integral equation governing the pressure distribution along the internal line support. Particularly, the singularities of the solution were made explicitly by means of a finite Hilbert transform. And the regular part was governed by a Fredholm integral of the second kind. It should be noted that the pressure distribution exerted along the line of internal support was found to be an inverse – square – root singularity. Therefore, it has been indicated that the pressure distribution is integrable. This was contradicted the works of (Williams 1952), who firstly pointed out that a moment singularity should exist outside of the support region and, moreover, the Kirchhoff shear force along the support region is not integrable.

The work of (Keer and Stahl 1972) has treated the eigenvalue problem of rectangular plates with partially clamped and simply supported along one edge using a modification technique presented by (Westmann and Yang 1967). The final results were obtained in terms of the homogeneous Fredholm integral equations of the second kind for an unknown auxiliary function. Further investigations have been made in (Stahl and Keer 1972a) and (Stahl and Keer 1972b) involving vibration and stability of rectangular plates with cracks and an internal lined – support. The mentioned works has pointed out that the moment singularities are of inverse – square – root type. And they occur at the transition points between two different supports, the vicinity of the crack tips, and the ends of partial internal line support. These singularities have been included explicitly in their detailed formulations.

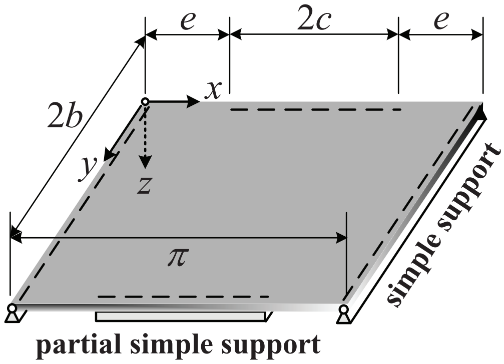

Figure 1: Geometry and Scaled Dimensions of Plate

After those works reviewed, this present study is mainly focused on the application of finite Hankel integral transform on analytical analysis of transversely free vibration of rectangular plate. The emphasis is placed on the analysis and formulation of double symmetrical free vibrating of such plate with mixed edge support conditions.

Figure 1 shows geometry of thin rectangular plate with actual length, $\bar{a}$. For the problem simplicity, this plate is scaled by the factor $\pi / \bar{a}$ so that the scaled length of the plate becomes π . It has fully simply supported edges along the lines which $x = 0$ and $x = \pi$, while the others are mixed between simply supported and free on the edges where $y = 0$ and $y = 2b$. On the mixed boundaries, the symmetrically simple supports of the length, $2c$ are aligned center while the rest are free with length, e . Therefore, the conditions along these two latter edges are mixed between simple and free support conditions.

It has been concluded in (Williams 1952) that at the points of transition from a simple support to a free edge, the moment singularities are in the order of $O(\varepsilon^{-1/2})$ while shear singularities are in $O(\varepsilon^{-3/2})$ if ε representing a small distance from those adjacent points. This is indicated that the Kirchhoff shear becomes nonintegrable.

2. Formulation

From the elementary plate theory (Timoshenko and Woinowsky – Krieger 1959), the fourth – order partial differential equation of motion can be expressed for the undamped free vibration case as

$$D\nabla^4 w_o + \mu \frac{\partial^2 w_o}{\partial t^2} = 0 \quad (1)$$

where $w_o(x, y, t)$ is transverse displacement, D is the flexural rigidity and μ is the mass densities per unit area. Equation (1) is for the uniform thickness plate with viscous damping. In order to formulate the eigenvalue problem, $w_o(x, y, t)$ may take the form

$$w_o(x, y, t) = w(x, y) f(t) = w(x, y)\left\{\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t)\right\} \quad (2)$$

where $w(x, y)$ is transverse displacement function which depends only on the spatial (position) coordinates (x, y) . And $f(t)$ is defined to be a time – dependent harmonic function of circular frequency (ω). Moreover, it is given that $i = \sqrt{-1}$ and ∇^2 is Laplacian operator. Then, equation (1) can be rewritten as

$$\left(\nabla^4 - k^2\omega^2\right) w\left(x, y\right) = 0 \quad (3)$$

where $k^2 = \mu / D$. Due to the fully simply supported on the opposite edges, $w(x, y)$ can be written immediately in the form of Levy – Nadai approach which gives

$$w(x, y) = \sum_{m=1,2,3,...}^{\infty} Y_m(y) \sin(mx) \quad (4)$$

According to the works of (Keer and Stahl 1972), the general solution of equation (4) is given by

$$Y_m(y) = \frac{1}{D}\left[A_m \sinh(r_1 y) + B_m \cosh(r_1 y) + C_m \sinh(r_2 y) + D_m \cosh(r_2 y)\right] \quad (5)$$

where $r_1 = (k\omega + m^2)^{1/2}$ and $r_2 = (-k\omega + m^2)^{1/2}$ and the constants A_m, B_m, C_m, and D_m can be determined from the prescribed boundary conditions of the plate. They are finally found to be

$$A_m = \eta' \tanh\left(r_1 b\right) D_m,\ B_m = -\eta' D_m,\ C_m = -\tanh\left(r_2 b\right) D_m \tag{6}$$

where η' is given by

$$\eta' = -\frac{k\omega - (1-\nu)m^2}{k\omega + (1-\nu)m^2} \tag{7}$$

Then, the remaining boundary conditions along mixed edges can be written as the dual – series equations, which are

$$\sum_{m=1,3,5,\ldots}^{\infty} mP_m\left[1 + F_m(\omega)\right]\sin(mx) = 0, \quad 0 \le x < e \tag{8}$$

$$\sum_{m=1,3,5,\ldots}^{\infty} P_m \sin(mx) = 0, \quad e < x \le \frac{\pi}{2} \tag{9}$$

where

$$P_m = \frac{2k\omega m^2}{k\omega + (1-\nu)m^2} D_m \tag{10}$$

$$\begin{aligned} 1 + F_m(\omega) = {} & \left[m^3 k\omega(1-\nu)(3+\nu)\right]^{-1} \\ & \times\left\{r_2 \tanh(r_2 b)\left[k\omega + (1-\nu)m^2\right]^2 - r_1 \tanh(r_1 b)\left[k\omega - (1-\nu)m^2\right]^2\right\} \end{aligned} \tag{11}$$

By applying the binomial expansion theorem (Kreyzig 2006) onto r_1 and r_2 resulting that

$$r_1 = m\left[1 + \frac{k\omega}{2m^2} - O(m^{-4})\right] \tag{12}$$

$$r_2 = m\left[1 - \frac{k\omega}{2m^2} - O(m^{-4})\right] \tag{13}$$

It is notable that $r_{1,2} \sim m$ when $m \to \infty$. After back substituting equation (12) and (13) into equation (11) for the expression of $F_m(\omega)$ and the property; $\tanh(\infty) = 1$, one can achieve the asymptotic form of $F_m(\omega)$ as shown below,

$$F_m(\omega) \sim -\frac{(k\omega)^2}{(1-\nu)(3+\nu)m^4} \tag{14}$$

This reveals that the function $F_m(\omega)$ approaches zero in the order of $O(m^{-4})$ as the integer m goes infinity. Furthermore, the constant that found in the bracket term of equation (8) will be served to isolate the singularity corresponding to the points of transition between simple support and free edge. This is one of the advantages of this approach. In the present stage, the solution of problem is then reduced to the determination of a single unknown function P_m written in terms of the unknown constant D_m as seen in equation (10).

3. Hankel Integral Transform Method

To determine the unknown function P_m analytically, the solution to the dual – series equations proceeds by using a finite Hankel integral transform. With this method, those two equations may be reduced to a single integral equation representing the unknown function P_m to be determined. Therefore, P_m is selected to satisfy equation (9) which gives.

$$P_m = \int_0^e t\varphi(t)\left[J_1(mt) - \frac{t}{e}J_1(me)\right]dt; m = 1,3,5,... \tag{15}$$

in which t is a dummy variable, $\varphi(t)$ is the introduced unknown auxiliary function to be determined and $J_n(u)$ is the Bessel function of the first kind of order n with argument u. By substitution of P_m in equation (15) into equation (9) and interchanging the order of summation and apply some identities, one obtains

$$\int_0^e t\varphi(t)\left[\frac{xH(t-x)}{2t(t^2-x^2)^{1/2}} - \frac{xtH(e-x)}{2e^2(e^2-x^2)^{1/2}}\right]dt = 0 \tag{16}$$

It should be observed that all terms in the bracket of equation (18) are vanished, because x is always larger than t and e; $e < x \le \pi/2$ as seen in equation (9) and $0 \le t \le e$. Hence, the Heaviside's functions $H(t-x)$ and $H(e-x)$ are equally zero.

The last task is now to reduce a remaining equation in equation (8) to the form of single integral equation. Thus, integrating equation (8) once with respect to x, and substituting P_m from equation (15) together with changing the order of summation and integration, an integral equation of Abel – type can be found as follows

$$\begin{aligned} h(x) &= \int_0^x \frac{x\varphi(t)}{\sqrt{x^2-t^2}}dt \\ &= e\int_0^1 \varphi(er)\left\{1 - r^2 + 2er\int_0^\infty \left[\exp(\pi s)+1\right]^{-1} \times \left[I_1(ser) - rI_1(se)\right]\cosh(xs)ds\right\}dr \\ &\quad + 2e^2\int_0^1 r\varphi(er)\sum_{m=1,3,5,...}^{\infty}\left[J_1(mer) - rJ_1(me)\right]\times F_m(\omega)\cos(mx)dr; \quad 0 \le x < e \end{aligned} \tag{17}$$

where r is a new dummy variable and $0 \le r \le 1$, and $I_n(u)$ is the modified Bessel function of the first kind of order n with argument u. Finally, a general solution of equation (17) becomes

$$\varphi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{h(x)}{\sqrt{t^2-x^2}}dx; 0 \le t < e \tag{18}$$

By substitution equation (17) into the above term and changing the order of integration, yields

$$\varphi(t)=e\int_0^1\varphi(er)\left(1-r^2\right)dr\left(\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{dx}{\sqrt{t^2-x^2}}\right)$$
$$+2e^2\int_0^1 r\varphi(er)\int_0^\infty\left[\exp(\pi s)+1\right]^{-1}\times\left[I_1(ser)-rI_1(se)\right]ds\,dr\left(\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{\cosh(sx)dx}{\sqrt{t^2-x^2}}\right) \quad (19)$$
$$+2e^2\int_0^1 r\varphi(er)\sum_{m=1,3,5,...}^{\infty}F_m(\omega)\left[J_1(mer)-rJ_1(me)\right]dr\times\left(\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{\cos(mx)dx}{\sqrt{t^2-x^2}}\right);\ 0\le t<e$$

With the help of certain identities found in (Gradshteyn and Ryzhik 1956) and changing the variable t to $e\rho$ where $0\le\rho\le 1$, the final result can be obtained and it appears to be

$$\Phi(\rho)+\int_0^1 K(\rho,r)\Phi(r)\,dr=0 \quad (20)$$

in which $\Phi(\rho)=\varphi(e\rho)$ and $\Phi(r)=\varphi(er)$. Consequently, the kernel function, $K(\rho,r)$ is given by

$$K(\rho,r)=2e^2r\left\{\sum_{m=1,3,5,...}^{\infty}m\left[J_1(mer)-rJ_1(me)\right]\times F_m(\omega)J_1(me\rho)-\int_0^\infty\frac{s\left[I_1(ser)-rI_1(se)\right]I_1(se\rho)}{\exp(\pi s)+1}ds\right\} \quad (21)$$

Noted that the form of integral equation presented in equation (20) is generally known to be a homogeneous Fredholm integral equation of the second kind with the unknown auxiliary function $\Phi(\rho)$. At this view point, the equation governed to the problem considered becomes a single integral equation instead of the partial differential equation. This is the main advantage of the present method for analyzing the mixed boundary value problems with higher – order partial differential equations.

4. Singularity Verification

Since the problem has mixed edge conditions, so it is expected to have singularities of the bending fields at the connecting point between the simple support and free edge. (Williams 1952) postulated that the moment singularity occurred at such point is of the order $O(\varepsilon^{-1/2})$, while one belong to the Kirchhoff shearing forces is in the order of $O(\varepsilon^{-3/2})$. Thus, the purpose of this section is to derive the proper singularity of problem that emerged in the solution. This is said to be another advantage of the finite Hankel integral transform method developed here.

The Kirchhoff shearing force or the reaction exerted by a partial simple support can be computed by the ordinary term as in (Timoshenko and Woinowsky 1959). After back substituting the expressions in equations (15) and (19) into that term, one gets

$$V_y(x,0) = \frac{(1-\nu)(3+\nu)\pi^3}{2\bar{a}^3}\left\{\left[-\int_0^e \frac{t^2}{e}\varphi(t)dt\right]\times\left(\frac{d}{dx}\left[\frac{1}{2e}-\frac{x}{2e}(x^2-e^2)^{-1/2}\right]\right.\right.$$
$$\left.+\frac{d}{dx}\left[\int_0^\infty\left[\exp(\pi s)+1\right]^{-1} I_1(ts)\cosh(xs)ds\right]\right) \tag{22}$$
$$\left.+\frac{d}{dx}\sum_{m=1,3,5,...}^{\infty}\int_0^t t\varphi(t)J_1(mt)\cos(mx)dt+\frac{d}{dx}\sum_{m=1,3,5,...}^{\infty}P_m F_m(\omega)\cos(mx)\right\}$$

It is clearly observed that only the first differentiate term contributes to the singularity in the shearing force. Letting $x = e+\varepsilon$ and taking binomial expansion theorem, equation (22) becomes

$$V_y(e+\varepsilon,0) = \left[\frac{(1-\nu)(3+\nu)\pi^3}{4\bar{a}^3}\int_0^e t^2\varphi(t)dt\right](2e\varepsilon)^{-3/2} + O(\varepsilon^{-1/2}) + O(\varepsilon^{1/2}) \tag{23}$$

This is completely demonstrated that $V_y(e+\varepsilon,0)\to\infty$ in the order of $O(\varepsilon^{-3/2})$ as $\varepsilon\to 0$ correspondence to $x\to e$ if the function P_m is assumed in the form of equation (15).

5. Discussion and Conclusion

In the foregoing analysis, it demonstrated that the transverse displacement function for free vibration problem of rectangular thin plate having partially simply supported edges can be formulated in the form of a single Fourier series by using the Levy – Nadai approach. The mixed boundary conditions that resulted from the partial simple support are then written as dual – series equations with a weight function. And they are explicitly reduced to the final form of a homogeneous Fredholm integral equation of the second kind. Furthermore, the nature of singularities that are allowed in the fields have verified.

Because of the complexity of the kernel, the integral equation; equation (20), can be treated numerically by considering the values for unknown auxiliary function $\Phi(\rho)$ over a finite number of equally spaced points using Simpson's rule and establishing a matrix equation for a system of linear homogeneous algebraic equations that is equivalent to the integral equation. A sufficient number of points should be chosen in order to ensure the accuracy of the solution, and the resulting matrix equation can be solved using a digital computer within the desired degree of highest accuracy. In addition, the integrand of the infinite integral in the kernel is found to be a monotonically increasing function up to some maximum value. After that it will be decayed exponentially. With these characteristics of the integrand, the improper infinite integral can be numerically evaluated using the appropriate Gauss quadrature methods such as Gauss – Legendre formula or Gauss – Laguerre formula.

However, the numerical results are not evaluated and presented here, because the objectives of the present paper are to propose the developed analytical method for analyzing the titled problem, and to show the correct moment singularities that are in the order of an inverse – square – root type. These have done completely as demonstrated in previous sections.

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References

[1] Nowacki W and Olesiak Z (1956).Vibration, buckling and bending of a circular plate clamped along part of its periphery and simply supported on the remaining part. Bulletin De L' Academic Polonaise Des Sciences, 4(4), pp. 247 – 258.

[2] Gladwell GML (1958). Some mixed boundary value problems in isotropic thin plate theory", Quart. J. Mech. Appl. Math. 11, pp. 159-171.

[3] Zorski H (1958). Plates with discontinuous supports", Archiwum Mechanika Stosowanej.(10), pp. 271 – 313.

[4] Bartlett CC (1963). The vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder. Quart. J. Mech. Appl. Math. 16, pp. 431 – 440

[5] Noble B (1965). The vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder. Proceeding of the Ninth Midwestern Mechanics Conference, 3(2), pp. 141 – 146.

[6] Yang WH (1968). On an integral equation solution for a plate with internal support. Quart. J. Mech. Appl. Math. 21, pp. 503 – 513.

[7] Williams ML (1952). Surface stress singularities resulting from various boundary conditions in angular corners of plates under bending. Proceedings of the 1st U.S. National Congress of Applied Mechanics, ASME, pp. 325 – 329.

[8] Keer LM and Stahl B (1972). Eigenvalue problems of rectangular plates with mixed edge conditions.J. Appl. Mech. 39, pp. 513 – 520.

[9] Westmann RA and Yang WH (1967). Stress analysis of cracked rectangular beams. J. Appl. Mech. 34, pp. 693 – 701.

[10] Stahl B and Keer LM (1972a). Vibration and stability of cracked rectangular plates. Int. J. Solids Struct. 8, pp. 69 – 91.

[11] Stahl B and Keer LM (1972b). Vibration and buckling of a rectangular plate with an internal support. Quart. J. Mech. Appl. Math. 25, pp. 467 – 478.

[12] Timoshenko SP and Woinowsky – Krieger S (1959). Theory of Plates and Shells, 2nd ed., McGraw – Hill, Singapore.

[13] Kreyszig E (2006). Advanced Engineering Mathematics, 9th ed., John Wiley & Sons, Inc., Singapore.

[14] Gradshteyn IN and Ryzhik IM (1956). Table of Integrals, Series, and Products, 4th ed., Academic Press, New York.